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Solvability of Fractionl q-Difference Equations of Order $2<\alpha\le 3$ Involving The P-Laplacian Operator

Yawen Yan   Chengmin Hou

(*Department of mathematics,Yanbian University,Yanji 133002,P.R.China*)

ABSTRACT : *In this paper, we study the existence of solutions for non-linear fractional q-difference equations of order* $2<\alpha\le 3$ *involving the p-Laplacian operator with various boundary value conditions. By using the Banach contraction mapping principle, we prove that, under certain conditions, the suggested non-linear fractional boundary value problem involving the p-Laplacian operator has a unique solution. Finally, we illustrate our results with some examples.*

KEYWORDS*: p-Laplacian operators; Caputo fractional derivative; Caputo fractional boundary value problem*

I. INTRODUCTION

In recent years, the topic of q-calculus has attracted the attention of several researchers and a variety of new results can be found in the papers [2-10] and the references cited therein. In 2010, Ferreira [1] considered the existence of nontrivial solutions to the fractional q-difference equation

$$\begin{aligned}&D_{q,0^+}^{\alpha}x(t)=f(t,x(t)),\quad 0\le t\le 1,\\&x(0)=x(1)=0.\end{aligned}$$

In [15], Aktuğlu and Özarslan dealt with the following Caputo q-fractional boundary value problem involving the p-Laplacian operator:

$$\begin{aligned}&D_q(\varphi_p({}^cD_q^{\alpha}x(t)))=f(t,x(t)),\ \ 0<t<1,\\&D_q^k x(0)=0,\ \ k=2,3,\ldots,n-1,\qquad x(0)=a_0x(1),\qquad D_qx(0)=a_1D_qx(1),\end{aligned}$$

where $a_0,a_1\neq 0$, $\alpha>1$, and $f\in C([0,1]\times R,R)$. Under some conditions, the authors obtained the existence and uniqueness of the solution for the above boundary value problem by using the Banach contraction mapping principle.

In this paper, we focus on the solvability of the following non-linear fractional differential equations of order $\alpha\in(2,3]$ involving the p-Laplacian operator with boundary conditions:

$$\begin{cases}D_q(\varphi_p({}^cD_q^{\alpha}x(t)))=f(t,x(t)),\\ {}^cD_q^{\alpha}x(0)=0,\\ x(0)=a_0x(1),\\ D_qx(0)=a_1D_qx(1),\\ D_q^2x(0)=a_2D_q^2x(1),\end{cases}$$

where $f \in C([0,1]\times R, R)$ and $x(t) \in C^2([0,1]\times R, R)$, $\phi_p(s)=|s|^{p-2}s$, $p>1$, $\varphi_p^{-1}=\varphi_v$, $\frac{1}{p}+\frac{1}{v}=1$,

${}^cD_q^\alpha$ is the fractional q-derivative of the Caputo type, D_q, D_q^2 is the fractional q-derivative and $a_0 \neq 1, a_1 \neq 1, a_2 \neq \frac{v+1}{2}, t \in [0,1], D^\alpha, D^\beta$ is the conformable fractional derivative.

2 Preliminaries on fractional *q*-calculus

In this section, we present basic definitions and results that will be needed in the rest of the paper. More detailed information about the theory of fractional q-calculus can be found in [6,7,8].

Let $q \in (0,1)$ and define

$$[a]_q := \frac{1-q^a}{1-q}, a \in R.$$

The q-analogue of the power function $(a-b)^{(n)}$ with $n \in N_0 := [0,1,2,\cdots]$ is

$$(a-b)^{(0)} := 1, \qquad (a-b)^{(n)} := \prod_{k=0}^{n-1}\left(a-bq^k\right), \; a,b \in R.$$

More generally, if $\alpha \in R$, then

$$(a-b)^{(\alpha)} := a^\alpha \prod_{n=0}^{\infty}\frac{1-(b/a)q^n}{1-(b/a)q^{\alpha+n}}, a \neq 0.$$

In particular, if $b=0$ then $a^{(\alpha)}=a^\alpha$. We also use the notation $0^{(\alpha)}=0^\alpha$ for $\alpha>0$. The q-gamma function is defined by

$$\Gamma_q(x) := \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}, x \in R\setminus\{0,-1,-2,\ldots\},$$

and satisfies $\Gamma_q(x+1)=[x]_q\Gamma_q(x)$.

Remark 2 ([14]) If $\alpha>0$ and $a \le b \le t$, then $(t-a)^{(\alpha)} \ge (t-b)^{(\alpha)}$.

It is well known that the beta function $B_q(t,s)$ has the following integral representation:for any $t,s>0$,

$$\begin{aligned} B_q(t,s) &= \int_0^1 \tau^{(t-1)}(1-q\tau)^{(s-1)}d_q\tau \\ &= (1-q)\sum_{n=0}^{\infty} q^n(1-q^{n+1})^{(s-1)}(q^n)^{(t-1)}. \end{aligned}$$

Moreover, $B_q(t,s)$ can be expressed in terms of $\Gamma(t)$, the gamma function, as follows:

$$B_q(t,s)=\frac{\Gamma_q(t)\Gamma_q(s)}{\Gamma_q(t+s)}.$$

Definition 2.1 Let $\alpha \geq 0$ and f be a function defined on $[0,T]$. The fractional *q*-integral of the Remann-Liouville type is given by $(I_q^0 f)(t) = f(t)$ and

$$\begin{aligned}(I_q^\alpha f)(t) &= \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)} f(s)d_q s \\ &= \frac{t^\alpha(1-q)}{\Gamma_q(\alpha)}\sum_{n=0}^{\infty} q^n(1-q^{n+1})^{(\alpha-1)} f(tq^n).\end{aligned} \tag{2.1}$$

Definition 2.2 The fractional *q*-derivative of Riemann-Liouville type of order $\alpha \geq 0$ is defined by $(D_q^0 f)(t) = f(t)$ and

$$(D_q^\alpha f)(t) = D_q^l I_q^{l-\alpha} f(t),\ \alpha > 0, \tag{2.2}$$

where *l* is the smallest integer greater than or equal to *v*.

Definition 2.3 ([6]) The fractional *q*-derivative of the Caputo type of order $\alpha \geq 0$ is defined by

$$({}^cD_q^\alpha f)(t) = I_q^{\lceil\alpha\rceil-\alpha} D_q^{\lceil\alpha\rceil} f(t) = \frac{1}{\Gamma_q(n-\alpha)}\int_0^t (t-qs)^{n-\alpha-1}(D_q^n f)(s)d_q s, \alpha > 0, \tag{2.3}$$

where $\lceil\alpha\rceil$ is the smallest integer greater than or equal to α .

Recall that $C^n([a,b],R)$ is the space of all real-valued function $x(t)$ which have continuous derivatives up to order $n-1$ on $[a,b]$. In the following lemmas, we give some auxiliary results which will be used in the sequel.

Lemma 2.1 ([6]) Let $\alpha > 0$, then the following equality holds:

$$(I_q^\alpha {}^cD_q^\alpha f)(t) = f(t) - \sum_{k=0}^{\alpha-1}\frac{t^k}{\Gamma_q(k+1)}(D_q^k f)(0). \tag{2.4}$$

On the other hand, the operator $\phi_p(s) = |s|^{p-2} s$, where $p > 1$, is called the p-Laplacian operator. It is easy to see that $\varphi_p^{-1} = \varphi_v$, where $\frac{1}{p}+\frac{1}{v}=1$. The following properties of the *p*-Laplacian operator will play an important role in the rest of the paper.

Lemma 2.2 ([19])Let φ_p be a *p*-Lplacian operator.

(i) If $1 < p < 2$, $xy > 0$, and $|x|,|y| \geq m > 0$, then

$$|\varphi_p(x) - \varphi_p(y)| \leq (p-1)m^{p-2}|x-y|.$$

(ii) If $p \geq 2$, and $|x|,|y| \leq M$, then

$$|\varphi_p(x) - \varphi_p(y)| \leq (p-1)M^{p-2}|x-y|.$$

Lemma 2.3 Assume that $\alpha \in (2,3]$, $a_0 \neq 1$, $a_1 \neq 1$, $a_2 \neq \frac{q+1}{2}$, $t \in [0,1]$ and $h \in C([0,1])$. Then the solution $x(t)$ of the boundary value problem

$$\begin{cases} D_q(\varphi_p({}^cD_q^\alpha x(t))) = h(t), \\ {}^cD_q^\alpha x(0) = 0, \\ x(0) = a_0 x(1), \\ D_q x(0) = a_1 D_q x(1), \\ D_q^2 x(0) = a_2 D_q^2 x(1), \end{cases} \tag{2.6}$$

can be represented by the following integral equation:

$$\begin{aligned} x(t) = {} & \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-q\tau)^{(\alpha-1)}\varphi_v\left(\int_0^\tau h(s)ds\right)d\tau \\ & + \frac{A_0}{\Gamma_q(\alpha)}\int_0^1 (1-q\tau)^{(\alpha-1)}\varphi_v\left(\int_0^\tau h(s)ds\right)d\tau \\ & + \Phi(t)\int_0^1 (1-q\tau)^{(\alpha-2)}\varphi_v\left(\int_0^\tau h(s)ds\right)d\tau \\ & + \psi(t)\int_0^1 (1-q\tau)^{(\alpha-3)}\varphi_v\left(\int_0^\tau h(s)ds\right)d\tau, \end{aligned} \tag{2.7}$$

where $A_0 = \frac{a_0}{1-a_0}$, $A_1 = \frac{a_1}{1-a_1}$, $A_2 = \frac{a_2(1+q)}{1+q-2a_2}$, and $\Phi(t) = \frac{A_0 A_1 + A_1 t}{\Gamma_q(\alpha-1)}$,

$$\psi(t) = \frac{A_1 A_2 (A_0 + t) + A_2[A_0(1+q) + t^2]}{(1+q)\Gamma_q(\alpha-2)}.$$

Proof Using (2.4) and the fact that $\varphi_p({}^cD_q^\alpha x(0)) = 0$, we have

$$\varphi_p({}^cD_q^\alpha x(t)) = \int_0^t h(s)d_q s,$$

or equivalently,

$${}^cD_q^\alpha x(t) = \varphi_v\left(\int_0^t h(s)d_q s\right), \tag{2.8}$$

where $\frac{1}{p} + \frac{1}{v} = 1$. Applying the fractional integral operator I_q^α to both sides of (2.8), we get

$$x(t)-x(0)-D_qx(0)t-\frac{D_q^2x(0)}{\Gamma_q(3)}t^2=\frac{1}{\Gamma_q(\alpha)}\int_0^t(t-q\tau)^{(\alpha-1)}\varphi_v(\int_0^\tau h(s)d_qs)d_q\tau,$$

or equivalently,

$$x(t)=\frac{1}{\Gamma_q(\alpha)}\int_0^t(t-q\tau)^{(\alpha-1)}\varphi_v(\int_0^\tau h(s)d_qs)d_q\tau+x(0)+D_qx(0)t+\frac{D_q^2x(0)}{1+q}t^2, \quad (2.9)$$

$$D_qx(t)=\frac{1}{\Gamma_q(\alpha-1)}\int_0^t(t-q\tau)^{(\alpha-2)}\varphi_v(\int_0^\tau h(s)d_qs)d_q\tau+D_qx(0)+\frac{D_q^2x(0)}{1+q}2t, \quad (2.10)$$

$$D_q^2x(t)=\frac{1}{\Gamma_q(\alpha-2)}\int_0^t(t-q\tau)^{(\alpha-3)}\varphi_v(\int_0^\tau h(s)d_qs)d_q\tau+\frac{2D_q^2x(0)}{1+q}. \qquad (2.11)$$

Taking $t=1$ on both sides of (2.9), (2.10) and (2.11), we have

$$x(1)=\frac{1}{\Gamma_q(\alpha)}\int_0^1(1-q\tau)^{(\alpha-1)}\varphi_v(\int_0^\tau h(s)d_qs)d_q\tau+x(0)+D_qx(0)+\frac{D_q^2x(0)}{1+q}, \qquad (2.12)$$

$$D_qx(1)=\frac{1}{\Gamma_q(\alpha-1)}\int_0^1(1-q\tau)^{(\alpha-2)}\varphi_v(\int_0^\tau h(s)d_qs)d_q\tau+D_qx(0)+\frac{2D_q^2x(0)}{1+q}, \qquad (2.13)$$

$$D_q^2x(1)=\frac{1}{\Gamma_q(\alpha-2)}\int_0^1(1-q\tau)^{(\alpha-3)}\varphi_v(\int_0^\tau h(s)d_qs)d_q\tau+\frac{2D_q^2x(0)}{1+q}. \qquad (2.14)$$

Using (2.12), (2.13), (2.14) and the boundary value conditions in (2.6) we can get that

$$D_q^2x(0)=\frac{A_2}{\Gamma_q(\alpha-2)}\int_0^1(1-q\tau)^{(\alpha-3)}\varphi_v(\int_0^\tau h(s)d_qs)d_q\tau, \qquad (2.15)$$

$$\begin{aligned}D_qx(0)=&\frac{A_1}{\Gamma_q(\alpha-1)}\int_0^1(1-q\tau)^{(\alpha-2)}\varphi_v(\int_0^\tau h(s)d_qs)d_q\tau\\&+\frac{2A_2}{1+q}\frac{A_1}{\Gamma_q(\alpha-2)}\int_0^1(1-q\tau)^{(\alpha-3)}\varphi_v(\int_0^\tau h(s)d_qs)d_q\tau,\end{aligned} \qquad (2.16)$$

and

$$\begin{aligned}x(0)=&\frac{A_0}{\Gamma_q(\alpha)}\int_0^1(1-q\tau)^{(\alpha-1)}\varphi_v(\int_0^\tau h(s)d_qs)d_q\tau\\&+\frac{A_0A_1}{\Gamma_q(\alpha-1)}\int_0^1(1-q\tau)^{(\alpha-2)}\varphi_v(\int_0^\tau h(s)d_qs)d_q\tau\\&+\Phi(t)\int_0^1(1-q\tau)^{(\alpha-3)}\varphi_v(\int_0^\tau h(s)d_qs)d_q\tau\\&+\psi(t)\int_0^1(1-q\tau)^{(\alpha-3)}\varphi_v(\int_0^\tau h(s)d_qs)d_q\tau.\end{aligned} \qquad (2.17)$$

Substituting (2.15), (2.16) and (2.17) into (2.9) gives (2.7) and this completes the proof. □

II. SOLVABILITY OF THE FRACTIONAL BOUNDARY VALUE PROBLEM

This section is devoted to the solvability of the fractional boundary value problem given in (2.5). First, we obtain conditions for existence and uniqueness of the solution $x(t)$ of the fractional boundary value problem given in (2.5). Then, each result obtained here is illustrated by examples. Recall that $C[0,1]$, the space of continuous functions on $[0,1]$ is a Banach space with the norm $\|x\| = \max_{t\in[0,1]}|x(t)|$. Now consider $T_i : C[0,1] \to C[0,1], i = 0,1,$ with

$$T_0x(t) := \varphi_v\left(\int_0^t f(s, x(s))d_q s\right)$$

and

$$T_1x(t) = \frac{1}{\Gamma_q(\alpha)}\int_0^t (t - q\tau)^{(\alpha-1)} x(\tau)d_q\tau + \frac{A_0}{\Gamma_q(\alpha)}\int_0^1 (1 - q\tau)^{(\alpha-1)} x(\tau)d_q\tau$$
$$+ \Phi(t)\int_0^1 (1 - q\tau)^{(\alpha-2)} x(\tau)d_q\tau + \psi(t)\int_0^1 (1 - q\tau)^{(\alpha-2)} x(\tau)d_q\tau$$

Then the operator $T : C[0,1] \to C[0,1]$, defined by $T = T_1 \circ T_0$, is continuous and compact.

Theorem 3.1 Suppose $1 < v < 2$, $a_0 \neq 1$, $a_1 \neq 1$, $a_2 \neq \frac{v+1}{2}$, $q \in (0,1)$ and the following conditions hold: $\exists\lambda > 0$, $0 < \delta < \frac{2}{2-v}$ and d with

$$\begin{aligned} d \le &\left(\lambda^{2-v}\Gamma_q(\delta(v-2)+2+\alpha)(1+q)\right) \\ &/\Big[(v-1)\Gamma_q(\delta(v-2)+2)\Big((1+|A_0|)(1+|A_1|\| \cdot [\delta(v-2)+1+\alpha]_q) \\ &+ 2|A_2|(|A_1|(|A_0|+1)+|A_0|(1+q)+2)[\delta(v-2)+1+\alpha]_q[\delta(v-2)+\alpha]_q\Big)\Big], \end{aligned} \tag{3.1}$$

such that

$$[\delta]_q \lambda t^{\delta-1} \le f(t, x) \quad \text{for any } (t, x) \in (0,1]\times R, \tag{3.2}$$

and

$$|f(t, x) - f(t, y)| \le d|x - y| \quad \text{for } t \in [0,1] \text{ and } x, y \in R. \tag{3.3}$$

Then boundary value problem (2.5) has a unique solution.

Proof Using inequality (3.3), we get

$$\lambda t^{\delta} \le \int_0^t f(s, x)ds \quad \text{for any } (t, x) \in (0,1]\times R.$$

By Lemma 2.2 (i) and (3.3), we have

$$
\begin{aligned}
&|T_0x(t)-T_0y(t)| \\
&\quad =\left|\varphi_v\left(\int_0^t f(s,x(s))d_qs\right)-\varphi_v\left(\int_0^t f(s,y(s))d_qs\right)\right| \\
&\quad \le (v-1)(\lambda t^{\delta})^{v-2}\left|\int_0^t f(s,x(s))d_qs-\int_0^t f(s,y(s))d_qs\right| \\
&\quad \le (v-1)\lambda^{v-2}t^{\delta(v-2)}\int_0^t |f(s,x(s))-f(s,y(s))|d_qs \\
&\quad \le d(v-1)\lambda^{v-2}t^{\delta(v-2)}\int_0^t |x(s)-y(s)|d_qs \\
&\quad \le d(v-1)\lambda^{v-2}t^{\delta(v-2)+1}\|x-y\|.
\end{aligned} \tag{3.4}
$$

Moreover,

$$
\begin{aligned}
&|Tx(t)-Ty(t)| \\
&\quad =|T_1(T_0x(t))-T_1(T_0y(t))| \\
&\quad =\left|\frac{1}{\Gamma_q(\alpha)}\int_0^t (t-q\tau)^{(\alpha-1)}\big((T_0x)(\tau)-(T_0y)(\tau)\big)d_q\tau\right. \\
&\qquad +\frac{A_0}{\Gamma_q(\alpha)}\int_0^1 (1-q\tau)^{(\alpha-1)}\big((T_0x)(\tau)-(T_0y)(\tau)\big)d_q\tau \\
&\qquad +\Phi(t)\int_0^1 (1-q\tau)^{(\alpha-2)}\big((T_0x)(\tau)-(T_0y)(\tau)\big)d_q\tau \\
&\qquad \left.+\psi(t)\int_0^1 (1-q\tau)^{(\alpha-2)}\big((T_0x)(\tau)-(T_0y)(\tau)\big)d_q\tau\right|.
\end{aligned} \tag{3.5}
$$

Finally, substituting (3.4) in (3.5), we get

$$
\begin{aligned}
&|Tx(t)-Ty(t)| \\
&\quad \le d(v-1)\lambda^{v-2}\|x-y\|\left[\frac{1}{\Gamma_q(\alpha)}\int_0^t (t-q\tau)^{(\alpha-1)}\tau^{\delta(v-2)+1}d_q\tau\right. \\
&\qquad +\left|\frac{A_0}{\Gamma_q(\alpha)}\right|\int_0^1 (1-q\tau)^{(\alpha-1)}\tau^{\delta(v-2)+1}d_q\tau \\
&\qquad +|\Phi(t)|\int_0^1 (1-q\tau)^{(\alpha-2)}\tau^{\delta(v-2)+1}d_q\tau \\
&\qquad \left.+|\psi(t)|\int_0^1 (1-q\tau)^{(\alpha-3)}\tau^{\delta(v-2)+1}d_q\tau\right].
\end{aligned} \tag{3.6}
$$

Using the equality

$$
\begin{aligned}
&\int_0^t (t-q\tau)^{(\alpha-1)}\tau^{\delta(v-2)+1}d_q\tau \\
&=t^{\alpha+\delta(v-2)+1}B_q(\delta(v-2)+2,\alpha)=t^{\alpha+\delta(v-2)+1}\frac{\Gamma_q(\delta(v-2)+2)\Gamma_q(\alpha)}{\Gamma_q(\delta(v-2)+2+\alpha)},
\end{aligned}
$$

in (3.6) one can write that

$$
\begin{aligned}
&|Tx(t)-Ty(t)| \\
&\quad \le d(v-1)\lambda^{v-2}\|x-y\|\left[\frac{t^{\alpha+\delta(v-2)+1}}{\Gamma_q(\alpha)}B(\delta(v-2)+2,\alpha)\right. \\
&\qquad +\left|\frac{A_0}{\Gamma_q(\alpha)}\right|B(\delta(v-2)+2,\alpha) \\
&\qquad \left.+|\Phi(t)|B(\delta(v-2)+2,\alpha-1)+|\psi(t)|\,B(\delta(v-2)+2,\alpha-2)\right]
\end{aligned}
$$

$$= d(v-1)\lambda^{v-2}\|x-y\|B(\delta(v-2)+2,\alpha)$$
$$\times\left[\frac{t^{\alpha+\delta(v-2)+1}}{\Gamma_q(\alpha)}+\frac{|A_0|}{\Gamma_q(\alpha)}+\frac{|A_1(A_0+t)|}{\Gamma_q(\alpha-1)}\cdot\frac{[\delta(v-2)+\alpha+1]_q}{[\alpha-1]_q}\right.$$
$$\left.+\frac{\left|2A_1A_2(A_0+t)+A_2\left(A_0(1+q)+2t^2\right)\right|}{(1+q)\Gamma_q(\alpha-2)}\cdot\frac{[\delta(v-2)+\alpha]_q[\delta(v-2)+\alpha+1]_q}{[\alpha-2]_q[\alpha-1]_q}\right],$$

using (1), we get that

$$|Tx(t)-Ty(t)|$$
$$\le d(v-1)\lambda^{v-2}\|x-y\|\frac{\Gamma_q(\delta(v-2)+2)\Gamma_q(\alpha)}{\Gamma_q(\delta(v-2)+2+\alpha)}$$
$$\times\left[\frac{1}{\Gamma_q(\alpha)}+\frac{|A_0|}{\Gamma_q(\alpha)}+\frac{|A_1(A_0+1)|}{\Gamma_q(\alpha)}\cdot[\delta(v-2)+\alpha+1]_q\right.$$
$$\left.+\frac{\left|2A_1A_2(A_0+1)+A_2\left(A_0(1+q)+2\right)\right|}{(1+q)\Gamma_q(\alpha)}\cdot[\delta(v-2)+\alpha+1]_q[\delta(v-2)+\alpha]_q\right]$$
$$\le d(v-1)\lambda^{v-2}\frac{\Gamma_q(\delta(v-2)+2)}{\Gamma_q(\delta(v-2)+2+\alpha)}\times\left[(1+|A_0|)(1+|A_1|\cdot[\delta(v-2)+1+\alpha]_q)\right.$$
$$\left.+\frac{2|A_2|\left(|A_1|(|A_0|+1)+|A_0|(1+q)+2\right)}{(1+q)}[\delta(v-2)+\alpha+1]_q[\delta(v-2)+\alpha]_q\right]\|x-y\|$$
$$=K\|x-y\|,$$

where

$$K=d(v-1)\lambda^{v-2}\frac{\Gamma_q(\delta(v-2)+2)}{\Gamma_q(\delta(v-2)+2+\alpha)}\times\left[(1+|A_0|)(1+|A_1|\cdot[\delta(v-2)+\alpha+1]_q)\right.$$
$$\left.+\frac{2|A_2|\left(|A_1|(|A_0|+1)+|A_0|(1+q)+2\right)}{(1+q)}[\delta(v-2)+\alpha+1]_q[\delta(v-2)+\alpha]_q\right],\quad (3.7)$$

Combining (3.7) with (3.1) implies that $0<K<1$, therefore T is a contraction. As a consequence of the Banach contraction mapping theorem [22] the boundary value problem given in (2.5) has a unique solution. □

Theorem 3.2 Suppose $1<v<2$, a_0, $a_1\neq 1$, $a_2\neq\frac{v+1}{2}$, $q\in(0,1)$ and the following conditions hold: $\exists\lambda>0$, $0<\delta<\frac{2}{2-v}$, and d satisfies (3.1).

such that

$$f(t,x)\le-[\delta]_q\lambda t^{\delta-1},\text{ for any }(t,x)\in(0,1]\times R.$$

and

$$|f(t,x)-f(t,y)|\le d|x-y|,\text{ for }t\in[0,1]\text{ and }x,y\in R.$$

Then boundary value problem (2.5) has a unique solution.

Proof The inequality $f(t,x) \le -\lambda\delta t^{\delta-1}$ implies that $\lambda\delta t^{\delta-1} \le -f(t,x)$. Therefore replace

$f(t,x)$ by $-f(t,x)$ in the proof of Theorem 1, we can get the proof of this theorem. □

Theorem 3.3 Suppose $v>2$, $a_0, a_1 \ne 1, a_2 \ne \frac{v+1}{2}$. There exists a non-negative function

$g(x) \in L[0,1]$, with $M := \int_0^1 g(\tau)d\tau \ge 0$ such that

$$|f(t,x)| \le g(t) \quad \text{for any } (t,x) \in (0,1] \times R, \tag{3.8}$$

and there exists a constant *d* with

$$\begin{aligned} d \le &\left(M^{2-v}\Gamma_q(2+\alpha)(1+q)\right) \\ &/\Big[(v-1)\Big((1+|A_0|)(1+|A_1|) \cdot [1+\alpha]_q\Big) \\ &+ 2|A_2|\big(|A_1|(|A_0|+1)+|A_0|(1+q)+2\big)[1+\alpha]_q[\alpha]_q\Big)\Big], \end{aligned} \tag{3.9}$$

and

$$|f(t,x) - f(t,y)| \le d|x-y| \quad \text{for } t \in [0,1] \text{ and } x, y \in R.$$

Then boundary value problem (2.5) has a unique solution.

Proof Using (3.8), we get that

$$\int_0^t |f(\tau, x(\tau))| d_q\tau \le \int_0^1 g(\tau)d\tau = M. \tag{3.10}$$

For all $t \in [0,1]$. By the definition of the operator T_0, one can write that

$$\begin{aligned} &|T_0x(t) - T_0y(t)| \\ &= \left|\varphi_v\left(\int_0^t f(s,x(s))d_qs\right) - \varphi_v\left(\int_0^t f(s,y(s))d_qs\right)\right|. \end{aligned} \tag{3.11}$$

As a consequence of (2.4), (3.10) and (3.11), we have

$$\begin{aligned} &|T_0x(t) - T_0y(t)| \\ &\le (v-1)M^{v-2}\left|\int_0^t f(s,x(s))d_qs - \int_0^t f(s,y(s))d_qs\right| \\ &\le (v-1)M^{v-2}\int_0^t |f(s,x(s)) - f(s,y(s))| d_qs \\ &\le d(v-1)M^{v-2}\int_0^t |x(s) - y(s)| d_qs \\ &\le (v-1)M^{v-2}t\|x-y\|. \end{aligned}$$

Moreover,

$$
\begin{aligned}
&|Tx(t)-Ty(t)|\\
&\quad=|T_1(T_0x(t))-T_1(T_0y(t))|\\
&\quad=\Bigg|\frac{1}{\Gamma_q(\alpha)}\int_0^t(t-q\tau)^{(\alpha-1)}((T_0x)(\tau)-(T_0y)(\tau))d_q\tau\\
&\qquad+\frac{A_0}{\Gamma_q(\alpha)}\int_0^1(1-q\tau)^{(\alpha-1)}((T_0x)(\tau)-(T_0y)(\tau))d_q\tau\\
&\qquad+\frac{A_0A_1+A_1t}{\Gamma_q(\alpha-1)}\int_0^1(1-q\tau)^{(\alpha-2)}((T_0x)(\tau)-(T_0y)(\tau))d_q\tau\\
&\qquad+\frac{2A_1A_2(A_0+t)+A_2[A_0(1+q)+2t^2]}{(1+q)\Gamma_q(\alpha-2)}\\
&\qquad\times\int_0^1(1-q\tau)^{(\alpha-3)}((T_0x)(\tau)-(T_0y)(\tau))d_q\tau\Bigg|.
\end{aligned}
$$

Since the equality

$$
\int_0^t(t-q\tau)^{(\alpha-1)}\tau d_q\tau=t^{\alpha+1}\int_0^1(1-q\tau)^{(\alpha-1)}\tau d_q\tau,
$$

we have

$$
\begin{aligned}
&|Tx(t)-Ty(t)|\\
&\quad\le d(v-1)M^{v-2}\|x-y\|\Bigg[\frac{t^{\alpha+1}}{\Gamma_q(\alpha)}\int_0^1(1-q\tau)^{(\alpha-1)}\tau d_q\tau+\frac{|A_0|}{\Gamma_q(\alpha)}\int_0^1(1-q\tau)^{(\alpha-1)}\tau d_q\tau\\
&\qquad+\frac{|A_1|(|A_0|+t)}{\Gamma_q(\alpha-1)}\int_0^1(1-q\tau)^{(\alpha-2)}\tau d_q\tau+\frac{2|A_2|[|A_1|(|A_0|+t)+(|A_0|(1+q)+2t^2)]}{(1+q)\Gamma_q(\alpha-2)}\\
&\qquad\times\int_0^1(1-q\tau)^{(\alpha-3)}\tau d_q\tau\Bigg]\\
&\quad\le d(v-1)M^{v-2}B_q(2,\alpha)\|x-y\|\times\Bigg[\frac{1}{\Gamma_q(\alpha)}+\frac{|A_0|}{\Gamma_q(\alpha)}+\frac{|A_1(A_0+1)|}{\Gamma_q(\alpha)}\cdot[\alpha+1]_q\\
&\qquad+\frac{|2A_2[A_1(A_0+1)+A_0(1+q)+2]|}{(1+q)\Gamma_q(\alpha)}[\alpha+1]_q[\alpha]_q\Bigg]\\
&\quad\le\frac{d(v-1)M^{v-2}}{\Gamma_q(\alpha+2)}\Big[(1+|A_0|)(1+|A_1|\cdot[\alpha+1]_q)\\
&\qquad+\frac{2|A_2|(|A_1|(|A_0|+1)+|A_0|(1+q)+2)}{(1+q)}[\alpha+1]_q[\alpha]_q\Big]\|x-y\|\\
&\quad\le K\|x-y\|,
\end{aligned}
$$

where

$$
\begin{aligned}
K=&\frac{d(v-1)M^{v-2}}{\Gamma_q(\alpha+2)}\Big[(1+|A_0|)(1+|A_1|\cdot[\alpha+1]_q)\\
&+\frac{2|A_2|(|A_1|(|A_0|+1)+|A_0|(1+q)+2)}{(1+q)}[\alpha+1]_q[\alpha]_q\Big].
\end{aligned}
$$

By (3.9), we get $K<1$, which implies that T is a contraction,therefore the boundary value problem given in (2.5) has a unique solution. □

In the present part, we illustrate our results by examples. We will use inequation in [18], one has

$\Gamma_q(t)\le\Gamma(t)$, for $0<t<1$ or $t\ge 2$; $\Gamma(t)\le\Gamma_q(t)$, for $1\le t\le 2$.

Example 1 Consider the following anti-periodic boundary value problem:

$$\begin{cases} D_q(\varphi_{\frac{7}{3}}({}^cD_q^{\frac{5}{2}}x(t)))=4t^2(2+\cos(\frac{\sqrt{\pi}x}{32}+\omega)), \quad t\in(0,1)\\ {}^cD_q^{\frac{5}{2}}x(0)=0,\\ x(0)=-x(1),\\ D_qx(0)=-D_qx(1), D_q^2x(0)=\frac{3}{10}D_q^2x(1), \end{cases} \tag{3.12}$$

where

$p=\frac{7}{3}, \alpha=\frac{5}{2}$, and $a_0=a_1=-1,\ a_2=\frac{3}{10}$.

Then

$v=\frac{7}{4}, |A_0|=|A_1|=|A_2|=\frac{1}{2}$, and take $\delta=4$, $\lambda=1$ and $d=\frac{\sqrt{\pi}}{8}$. because of $q\in(0,1)$,

so we let $q=\frac{1}{2}$, obviously,

$$\begin{aligned}&\big(\lambda^{2-v}\Gamma_q(\delta(v-2)+2+\alpha)(1+q)\big)/\{(v-1)\Gamma_q(\delta(v-2)+2)\times\\ &\quad[(1+|A_0|)(1+|A_1|)\cdot[\delta(v-2)+2+\alpha]_q)\\ &\quad+2|A_2|(|A_1|(|A_0|+1)+|A_0|(1+q)+2)[\delta(v-2)+1+\alpha]_q[\delta(v-2)+\alpha]_q]\}\\ &\ge\frac{1}{4}\Gamma_q(\frac{3}{2})\ge\frac{1}{4}\Gamma(\frac{3}{2})=\frac{\sqrt{\pi}}{8}=d,\end{aligned}$$

Moreover, it

$$\begin{aligned}K&=d(v-1)\lambda^{v-2}\frac{\Gamma_q(\delta(v-2)+2)}{\Gamma_q(\delta(v-2)+2+\alpha)}\times[(1+|A_0|)(1+|A_1|)\cdot[\delta(v-2)+\alpha+1]_q)\\ &\quad+\frac{2|A_2|(|A_1|(|A_0|+1)+|A_0|(1+q)+2)}{(1+q)}[\delta(v-2)+\alpha+1]_q[\delta(v-2)+\alpha]_q]\\ &\le\frac{55\sqrt{\pi}}{128\Gamma_q(3/2)}\le\frac{55\sqrt{\pi}}{128\Gamma(3/2)}=\frac{55}{64}<1.\end{aligned}$$

can be easy see that

$$[\delta]_q\lambda t^{\delta-1}\le[4]_qt^3\le 4t^2(2+\cos(\frac{\sqrt{\pi}x}{32}+\omega))=f(t,x).$$

Finally,

$$\begin{aligned}&|f(t,x)-f(t,y)|\\ &=\left|4t^2(2+\cos(\frac{\sqrt{\pi}}{32}x+\omega))-4t^2(2+\cos(\frac{\sqrt{\pi}}{32}y+\omega))\right|\\ &=4t^2\left|\cos(\frac{\sqrt{\pi}}{32}x+\omega)-\cos(\frac{\sqrt{\pi}}{32}y+\omega)\right|\\ &=4\left|(\frac{\sqrt{\pi}}{32}x+\omega)-(\frac{\sqrt{\pi}}{32}y+\omega)\right|\\ &=\frac{\sqrt{\pi}}{8}|x-y|.\end{aligned}$$

There fore as a consequence of Theorem 3.1, boundary value problem given in (3.12) has a unique solution.

Example 2 Consider the following boundary value problem:

$$\begin{cases} D_q(\varphi_{\frac{7}{4}}({}^cD_q^{\frac{5}{2}}x(t))) = \sin^2(\frac{\sqrt{\pi}x}{40}+\omega), \ \ t\in(0,1) \\ {}^cD_q^{\frac{5}{2}}x(0)=0,. \\ x(0)=-x(1), \\ D_qx(0)=-D_qx(1), D_q^2x(0)=\frac{3}{10}D_q^2x(1), \end{cases} \tag{3.13}$$

where

$p=\frac{7}{4}, \alpha=\frac{5}{2}$, and $a_0=a_1=-1, a_2=\frac{3}{10}$.

Then

$v=\frac{7}{3}, |A_0|=|A_1|=|A_2|=\frac{1}{2}$, and take $\lambda=1$ and $d=\frac{\sqrt{\pi}}{20}$. Because of $q\in(0,1)$, so

we let $q=\frac{1}{2}$, taking $g(t)=1$, we get $M=1$,

$$\begin{aligned} &\left(M^{2-v}\Gamma_q(2+\alpha)(1+q)\right)/\{(v-1)\times\left[(1+|A_0|)(1+|A_1|\cdot[2+\alpha]_q)\right. \\ &\quad \left. +2|A_2|(|A_1|(|A_0|+1)+|A_0|(1+q)+2)[1+\alpha]_q[\alpha]_q\right]\} \\ &\geq \frac{1}{20}\sqrt{\pi}=d, \end{aligned}$$

$$\begin{aligned} K &= \frac{d(v-1)M^{v-2}}{\Gamma_q(\alpha+2)}\left[(1+|A_0|)(1+|A_1|\cdot[\alpha+1]_q)\right. \\ &\quad \left.+\frac{2|A_2|(|A_1|(|A_0|+1)+|A_0|(1+q)+2)}{(1+q)}[\alpha+1]_q[\alpha]_q\right] \\ &\leq \frac{11}{27}<1. \end{aligned}$$

On the other hand,

$$\begin{aligned} &|f(t,x)-f(t,y)| \\ &\leq \left|\sin^2(\frac{\sqrt{\pi}}{40}x+\omega))-\sin^2(\frac{\sqrt{\pi}}{40}y+\omega))\right| \\ &\leq \frac{\sqrt{\pi}}{20}|x-y|. \end{aligned}$$

For $t\in[0,1]$ and $x,y\in R$.

Therefore by Theorem 3.3, the antiperiodic boundary value problem given in (3.13) has a unique solution.

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